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General recurrence relations for Clebsch–Gordan coefficients of the quantum algebra $U_q(su_2)$

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Abstract. Two general recurrence relations for the Clebsch-Gordan coefficients of the quantum algebra $U_q(su_2)$, connecting their values for different meanings of the parameter l defining the representations, are derived. Recurrence formulas for the Racah coefficients of this algebra are given. Recurrence relations for the Racah coefficients of $U_q(su_2)$ are easily obtained from those for the classical Racah coefficients.

1. Introduction

Quantum groups and algebras are of great importance for applications in classical and quantum integrable systems, in quantum field theory, in statistical physics and in the theory of basic hypergeometric functions. The quantum algebra $U_q(su_2)$ is intensively used in the conformal field theory. There are attempts (Iwao 1990, Raychev *et al* 1990) to explain experimental data of molecular, atomic and nuclear spectroscopy with the help of the representation theory for the quantum algebra $U_q(su_2)$. Recently Biedenharn (1989) and Macfarlane (1989) have considered a q-analogue of the quantum harmonic oscillator which is related to the algebra $U_q(su_2)$.

In order to apply the quantum algebra $U_q(su_2)$ in physics we need a well developed theory of its representations. It is important to have the good theory of Clebsch–Gordan (CG) and Racah coefficients. These coefficients are related to tensor products of representations. Tensor products are used for construction of the universal R-matrices which are of great importance for the Yang–Baxter equation.

Kirillov and Reshetikhin (1988) have initiated study into the CG and Racah coefficients of $U_q(su_2)$. Some three-terms recurrence relations for these coefficients were derived by Groza *et al* (1990), Kachurik and Klimyk (1990) and Nomura (1990). The present paper is devoted to derivation of general recurrence formulae for the CG coefficients. Many three-term recurrence relations (including new ones) can be obtained from these formulae. We also consider recurrence relations for the Racah coefficients of $U_q(su_2)$.

Let us note that recurrence relations for the CG and Racah coefficients are useful for evaluation of these coefficients, for studying their properties, for obtaining their numerical values, and for development of the theory of CG and Racah coefficients.

In section 2 we describe the quantum algebra $U_q(su_2)$ and give necessary information regarding the CG and Racah coefficients. In sections 3 and 4 we derive two general recurrence relations for the CG coefficients. Recurrence relations for the Racah coefficients are considered in section 5.

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2. The quantum algebra $U_q(su_2)$

The quantum algebra $U_q(su_2)$ is an associative algebra generated by the elements J_+, J_-, $J_z \equiv J$ obeying the commutation relations

$$[J, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = \frac{q^J - q^{-J}}{q^{1/2} - q^{-1/2}}$$

where q is a complex number. We suppose that q is not a root of unity.

As in the classical case, finite-dimensional representations T_l of the quantum algebra $U_q(su_2)$ are given by integral or half-integral non-negative number l. The orthonormal basis $|l, m\rangle$, $m = -l, -l+1, \ldots, l$, exists in the carrier space V_l of the representation T_l for which

$$J_\pm|l, m\rangle = ([l \mp m][l \pm m + 1])^{1/2}|l, m \pm 1\rangle \qquad J|l, m\rangle = m|l, m\rangle$$

where $[n]$ means the expression

$$[n] \equiv [n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = [n]_{q^{-1}}.$$

This expression is called a q-number.

The structure of a Hopf algebra is introduced into $U_q(su_2)$. According to this structure we have $T_l \otimes T_{l'} \neq T_{l'} \otimes T_l$. The representations $T_l \otimes T_{l'}$ and $T_{l'} \otimes T_l$ are related by the R-matrix. As in the classical case, the Clebsch–Gordan coefficients for the tensor product $T_{l_1} \otimes T_{l_2}$ of $U_q(su_2)$ are defined by the relation

$$|l_1, j\rangle|l_2, k\rangle = \sum_{l,m} C^{l_1 l_2 l}_{jkm}|l, m\rangle.$$

If $j + k \neq m$ then $C^{l_1 l_2 l}_{jkm} = 0$. The CG coefficients constitute the unitary matrix. The CG coefficients can be expressed in terms of the basic hypergeometric function ${}_3\Phi_2$. The function ${}_{n+1}\Phi_n$ is defined as

$$\begin{aligned}
{}_{n+1}\Phi_n &\left(\begin{matrix} a_1, a_2, \ldots, a_{n+1} \\ b_1, b_2, \ldots, b_n \end{matrix} \middle| q, z \right) \\
&\equiv {}_{n+1}\Phi_n(a_1, a_2, \ldots, a_{n+1}; b_1, b_2, \ldots, b_n; q, z) \\
&= \sum_r \frac{(q^{a_1}; q)_r \ldots (q^{a_{n+1}}; q)_r}{(q^{b_1}; q)_r \ldots (q^{b_n}; q)_r} \frac{z^r}{(q; q)_r}
\end{aligned}$$

where

$$(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j) \qquad n \neq 0$$

$$(a; q)_0 = 1 \qquad a \in C.$$

Below we shall use the q-binomial formula

$$(y - x)_q^n = y^n \left(1 - \frac{x}{y}\right)_q^n = \sum_{r=0}^{n} \frac{(q^{-n}; q)_r}{(q; q)_r} x^r y^{n-r}. \tag{1}$$

The formula

$$(1 - x)_q^n = \sum_{r=0}^{n} \frac{(q^{-n}; q)_r}{(q; q)_r} x^r = {}_1\Phi_0(-n; q, x) \tag{2}$$

is valid (Slater 1966).

The algebra $U_q(su_2)$ is a q-deformation of the universal enveloping algebra $U(su_2)$ of the classical Lie algebra su_2. It contains linear combinations of products of the elements J_+, J_-, J. The irreducible representation T_l of $U_q(su_2)$ in the basis $|l, m\rangle$ is given by the matrix with entries t^l_{mn} depending on elements $a \in U_q(su_2)$. Since $T_l \otimes T_{l'} \neq T_{l'} \otimes T_l$ then

$$t^l_{mn} t^{l'}_{ij} \neq t^{l'}_{ij} t^l_{mn}.$$

As in the classical case, the matrix elements and the CG coefficients are connected by the relation

$$\sum_{j,k,r,s} C^{l_1 l_2 l}_{jkm} C^{l_1 l_2 l}_{rsp} t^{l_1}_{jr} t^{l_2}_{ks} = t^l_{mp} \tag{3}$$

where $j+k=m$ and $r+s=p$ (Groza *et al* 1990).

Considering the tensor products $(T_{l_1} \otimes T_{l_2}) \otimes T_{l_3}$ and $T_{l_1} \otimes (T_{l_2} \otimes T_{l_3})$ we obtain the Racah coefficients $R(l_1 l_2 l_3, l_{12} l_{23}, l)$ which are related to the $6j$-symbols by the formula (see Kachurik and Klimyk 1990)

$$\begin{Bmatrix} l_1 & l_2 & l_{12} \\ l_3 & l & l_{23} \end{Bmatrix} = (-1)^{l_1+l_2+l_3+l}([2l_{12}+1][2l_{23}+1])^{-1/2} R(l_1 l_2 l_3, l_{12} l_{23}, l).$$

3. The first recurrence relation for Clebsch–Gordan coefficients

It is proved by Groza *et al* (1990) that

$$\begin{aligned}
(x-qy)^{a+b-c}_q (y-z)^{b+c-a}_q (z-x)^{a+c-b}_q
&= (-1)^{a+b-2c}[a-b+c]![b-a+c]! \\
&\quad \times \sum_{j+k=m} (-1)^{k-j} q^B \frac{x^{a+j} y^{b+k} z^{c-m}}{[a-j]![b+k]![c-a-k]![c-b+j]!} \\
&\quad \times {}_3\Phi_2(c-a-b, -a+j, -b-k; c-a-k+1, c-b+j+1; q, q)
\end{aligned} \tag{4}$$

where

$$B = \tfrac{1}{2}\{(c+1)(a+b-2c)+(b-a)(a-b+m)+(c+1)(k-j)\}$$

$$[m]! = [m][m-1]\ldots[1] = q^{-m(m-1)/4}(q;q)_m/(1-q)^m.$$

The relation

$${}_1\Phi_0(ab; q, x) = {}_1\Phi_0(a; q, x)\, {}_1\Phi_0(b; q, ax)$$

(Slater 1966) and the formulae (1) and (2) lead to the equality

$$(y-x)^{m+n}_q = (y-x)^m_q (y-q^{-m}x)^n_q. \tag{5}$$

Let $a = a' + a''$, $b = b' + b''$, $c = c' + c''$. Then according to formula (5) we have

$$\begin{aligned}
(x-qy)^{a+b-c}_q (y-z)^{b+c-a}_q (z-x)^{a+c-b}_q
&= (x-qy)^{a'+b'-c'}_q (y-z)^{b'+c'-a'}_q (z-x)^{a'+c'-b'}_q (x-q^{c'-a'-b'+1}y)^{a''+b''-c''}_q \\
&\quad \times (y-q^{a'-b'-c'}z)^{b''+c''-a''}_q (z-q^{b'-a'-c'}x)^{a''+c''-b''}_q.
\end{aligned} \tag{6}$$

Applying to the three last multipliers of the right-hand side of this formula the q-binomial decomposition (1) and repeating reasonings of the paper by Groza *et al* (1990), we obtain the relation

$$(x-q^{c'-a'-b'+1}y)_q^{a''+b''-c''}(y-q^{a'-b'-c'}z)_q^{b''+c''-a''}(z-q^{b'-a'-c'}x)_q^{a''+c''-b''}$$
$$=(-1)^{a''+b''-2c''}[a''+c''-b'']![b''+c''-a'']!$$
$$\times\sum_{j+k=m}\frac{(-1)^{k-j}q^{D}x^{a''+j}y^{b''+k}z^{c''-m}}{[a''-j]![b''+k]![c''-b''+j]![c''-a''-k]!}$$
$$\times{}_3\Phi_2\left(\begin{matrix}c''-a''-b'',j-a'',-k-b''\\c''-a''-k+1,c''-b''+j+1\end{matrix}\middle|\,q,q^{-a'-b'-c'+1}\right)\tag{7}$$

where

$$D=\tfrac{1}{2}\{(c''+1)(a''+b''-2c'')+(b''-a'')(a''-b''+m)+(c''+1)(k-j)\}$$
$$+c'(a''+b''-2c'')-(a'-b')(a''-b'')+(b'-a')m+c'(k-j).$$

Taking into account relations (4) and (7) we obtain from formula (6) that

$$V\,{}_3\Phi_2\left(\begin{matrix}c-a-b,-a+j,-b-k\\c-a-k+1,c-b+j+1\end{matrix}\middle|\,q,q\right)$$
$$=\sum_{\substack{j'+j''=j\\k'+k''=k}}q^{P}V'V''\,{}_3\Phi_2\left(\begin{matrix}c'-a'-b',-a'+j',-b'-k'\\c'-a'-k'+1,c'-b'+j'+1\end{matrix}\middle|\,q,q\right)$$
$$\times{}_3\Phi_2\left(\begin{matrix}c''-a''-b'',-a''+j'',-b''-k''\\c''-a''-k''+1,c''-b''+j''+1\end{matrix}\middle|\,q,q^{-a'-b'-c'+1}\right)\tag{8}$$

where

$$P=c'(a''+b''-2c''+k''-j'')+(b'-a')(a''-b''+m'')$$
$$V=q^{(c+1)(a+b-2c)/2+(c+1)(k-j)/2+(b-a)(a-b+m)/2}\tag{9}$$
$$\times[a-b+c]![b-a+c]!([a-j]![b+k]![(c-a-k]![c-b+j]!)^{-1}$$

and V', V'' are obtained from V with the help of replacements a by a' and a'', b by b' and b'' and so on.

Using the formula

$$C_{jkm}^{abc}=q^{H}\frac{\Delta(abc)([a+j]![b-k]![c-m]![c+m]![2c+1])^{1/2}}{[a+b-c]![c-a-k]![c-b+j]!([a-j]![b+k]!)^{1/2}}$$
$$\times{}_3\Phi_2(a-b-c,j-a,-k-b;c-a-k+1,c-b+j+1;q,q)$$

where

$$H=\tfrac{1}{4}(a+b-c)(a+b+c+1)+\tfrac{1}{2}(ak-bj)$$
$$\Delta(abc)=([a+b-c]![a-b+c]![b-a+c]!/[a+b+c+1]!)^{1/2}$$

we express ${}_3\Phi_2(\ldots;q,q)$ in terms of the CG coefficients and substitute into relation (8). After some simplifications we obtain the recurrence relation

$$UC_{jkm}^{abc}=\sum_{\substack{j'+j''=j\\k'+k''=k}}U'U''C_{j'k'm'}^{a'b'c'}{}_3\Phi_2\left(\begin{matrix}c''-a''-b'',-a''+j'',-k''-b''\\c''-a''-k''+1,c''-b''+j''+1\end{matrix}\middle|\,q,q^{-a'-b'-c'+1}\right)\tag{10}$$

where $m=j+k$, $m'=j'+k'$,

$$U=q^{-(3/4)(a+b-c)(a+b-c-1)-(1/2)(2c+1)(a+b)+2ab+(1/2)j(2b-a-c-1)}q^{-(1/2)k(2a-b-c-1)}$$
$$\times\left(\frac{[a+b-c]![a-b+c]![b-a+c]![a+b+c+1]!}{[a-j]![a+j]![b-k]![b+k]![c-m]![c+m]![2c+1]}\right)^{1/2}$$
$$U''=q^{c'(a''+b''-2c''+k''-j'')+(b'-a')(a''-b''+j''+k'')}V''$$

U' is obtained from U with the help of replacements of a by a', b by b' and so on.

Different three-term recurrence relations for the CG coefficients are special cases of (10). For example, for $a''=0$, $b''=\frac{1}{2}$, $c''=\frac{1}{2}$ we obtain

$$([2c][b-a+c][a+b+c+1]/[2c+1])^{1/2}C^{abc}_{jkm}$$
$$=q^{(b+c-m-k)/4}([b+k][c+m])^{1/2}C^{a,b-1/2,c-1/2}_{j,k-1/2,m-1/2}$$
$$+q^{-(b+c+m+k)/4}([b-k][c-m])^{1/2}C^{a,b-1/2,c-1/2}_{j,k+1/2,m+1/2}.$$

4. The second recurrence relation for Clebsch–Gordan coefficients

We write down relation (3) in the form

$$t^{l_1}_{j-j_2,j'+l_2}t^{l_2-k}_{j_2-k,k-l_2}=\sum_{l'=|l_1-l_2+k|}^{l_1+l_2-k}C^{l_1,l_2-k,l'}_{j-j_2,j_2-k,j-k}C^{l_1,l_2-k,l'}_{j'+l_2,k-l_2,j'+k}t^{l'}_{j-k,j'+k} \tag{11}$$

and multiply its both sides by

$$q^{-k(l_2-j_2)/2}([2l_2]![l_2+j_2-2k]!)^{1/2}([2l_2-2k]![l_2+j_2]!)^{-1/2}t^k_{k,-k}.$$

The explicit expressions for t^l_{ml} and $t^l_{m,-l}$ from the paper by Groza *et al* (1990) show that the relation

$$t^l_{m,-l}=q^{k(m-l)/2}\left(\frac{[2l]![l+m-2k]!}{[2l-2k]![l+m]!}\right)^{1/2}t^{l-k}_{m-k,k-l}t^k_{k,-k}$$

is valid. Using it in formula (11) we obtain the equality

$$t^{l_1}_{j-j_2,j'+l_2}t^{l_2}_{j_2,-l_2}=q^{-k(l_2-j_2)/2}\left(\frac{[2l_2]![l_2+j_2-2k]!}{[2l_2-2k]![l_2+j_2]!}\right)^{1/2}$$
$$\times\sum_{l'}C^{l_1,l_2-k,l'}_{j-j_2,j_2-k,j-k}C^{l_1,l_2-k,l'}_{j'+l_2,k-l_2,j'+k}t^{l'}_{j-k,j'+k}t^k_{k,-k}.$$

We apply to both its parts formula (11) and equate coefficients at $t^l_{jj'}$. This procedure is justified due to linear independence of matrix elements (Masuda *et al* 1988). It was shown by Groza *et al* (1990) that the CG coefficients $C^{l_1l_2l}_{j,-l,k}$ and $C^{l_1l_2l}_{jlk}$ are one-term expressions. Using these expressions we finally obtain the recurrence relation (which was independently received by V A Groza)

$$C^{l_1l_2l}_{j-j_2,j_2,j}=q^{(1/4)k(k+1)+(1/4)l(l+1)-(1/2)k(l_2+j_2+1)}$$
$$\times\left(\frac{[l_2+j_2-2k]![l+j]![l_1+l_2-l]![l_2-l_1+l]![l_1+l_2+l+1]![2l+1]}{[l_2+j_2]![l-j]![l_1-l_2+l]!}\right)^{1/2}$$
$$\times\sum_{l'=l-k}^{l+k}\frac{(-1)^{l'-l+k}q^{-l'(l'+1)/4}[2k]![l+l'-k]!}{[l-l'+k]![l+l'+k+1]![l'-l+k]!}C^{l_1,l_2-k,l'}_{j-j_2,j_2-k,j-k}$$
$$\times\left(\frac{[l'-j+k]![l_1-l_2+l'+k]![2l'+1]}{[l'+j-k]![l_2-l_1+l'-k]![l_1+l_2-l'-k]![l_1+l_2+l'-k+1]!}\right)^{1/2}. \tag{12}$$

The number of summands here is equal to $2k+1$. However, part of them vanishes. The non-negative integer k in (12) has to satisfy the condition $0 \leqslant k \leqslant (l_2+j_2)/2$. For $k=\frac{1}{2}$ we obtain from (12) the three-term recurrence relation

$$C^{l_1 l_2 l}_{m j_2 j} = q^{(l-l_2-m)/4}\left(\frac{[l+j][l_2-l_1+l][l_1+l_2+l+1]}{[2l+1][2l][l_2+j_2]}\right)^{1/2} C^{l_1, l_2-1/2, l-1/2}_{m, j_2-1/2, j-1/2}$$
$$- q^{-(l+l_2+m+1)/4}\left(\frac{[l-j+1][l_1+l_2-l][l_1-l_2+l+1]}{[l_2+j_2][2l+1][2l+2]}\right)^{1/2} C^{l_1, l_2-1/2, l+1/2}_{m, j_2-1/2, j-1/2}.$$

5. Recurrence relations for Racah coefficients

Recurrence relations for the Racah coefficients of the quantum algebra $U_q(su_2)$ can be derived from the q-analogue of the Biedenharn–Elliott identity by using special values of the parameters (Kachurik and Klimyk 1990, Nomura 1990). Other recurrence relations can be obtained from those for the function ${}_4\Phi_3$ since the Racah coefficients are expressed in terms of this function (Kirillov and Reshetikhin 1988, Kachurik and Klimyk 1990).

The recurrence relations

$$[a-b][e]\,{}_4\Phi_3(a,b,c,d;e,f,h;q,x)$$
$$= [a-e][b]\,{}_4\Phi_3(a,b+1,c,d;e+1,f,h;q,x)$$
$$-[b-e][a]\,{}_4\Phi_3(a+1,b,c,d;e+1,f,h;q,x) \qquad (13)$$

$$[a-1][f-e]\,{}_4\Phi_3(a,b,c,d;e,f,h;q,x)$$
$$= [f-1][a-e]\,{}_4\Phi_3(a-1,b,c,d;e,f-1,h;q,x)$$
$$-[e-1][a-f]\,{}_4\Phi_3(a-1,b,c,d;e-1,f,h;q,x) \qquad (14)$$

$${}_4\Phi_3(a,b,c,d;e,f,h;q,x)$$
$$= {}_4\Phi_3(a-1,b+1,c,d;e,f,h;q,x) - ([a-b][c][d]/[e][f][h])$$
$$\times {}_4\Phi_3(a,b+1,c+1,d+1;e+1,f+1,h+1;q,x) \qquad (15)$$

are valid for ${}_4\Phi_3$, where $e+f+h=a+b+c+d+1$. They are proved by comparing coefficients at the same powers of the variable x. Putting $x=q$ in (13)–(15) and using different expressions for the Racah coefficients in terms of ${}_4\Phi_3$, we can obtain many three-term recurrence relations for the Racah coefficients and for $6j$-symbols of the algebra $U_q(su_2)$. For example, we have

$$[b-c+e+f+1]([a+b+c+1][a+f-e])^{1/2}\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$$
$$= ([a-b+c][e+f-a+1][d+c-e]$$
$$\times [d-c+e+1])^{1/2}\begin{Bmatrix} a-\frac{1}{2} & b & c-\frac{1}{2} \\ d & e+\frac{1}{2} & f \end{Bmatrix}$$
$$-([a+b-c][a+e+f+1][b+d+f+1]$$
$$\times [b-d+f])^{1/2}\begin{Bmatrix} a-\frac{1}{2} & b-\frac{1}{2} & c \\ d & e & f-\frac{1}{2} \end{Bmatrix}$$

$$
([b-a+c+1][a-b+c][a+f-e][b+d-f+1])^{1/2}\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}
$$
$$
=([b+d+f+2][c+d+e+2][d-c+e+1]
$$
$$
\times[e+f-a+1])^{1/2}\begin{Bmatrix} a-\frac{1}{2} & b+\frac{1}{2} & c \\ d+\frac{1}{2} & e+\frac{1}{2} & f \end{Bmatrix}
$$
$$
+[b-a+d+e+1]([a+e+f+1][d-b+f])^{1/2}\begin{Bmatrix} a-\frac{1}{2} & b+\frac{1}{2} & c \\ d & e & f-\frac{1}{2} \end{Bmatrix}.
$$

In fact, we have obtained q-analogues of all known recurrence relations for the classical Racah coefficients (and for classical $6j$-symbols). Comparing classical recurrence relations with the recurrence formulae obtained for $U_q(su_2)$ we can make the following conclusion. Recurrence relations for the Racah coefficients of $U_q(su_2)$ are obtained from those for the classical Racah coefficients by replacement of all factorials $m!$ by q-factorials $[m]!$. Let us note that expressions for the Racah coefficients are obtained by using the same replacement from those for the classical Racah coefficients.

6. Conclusion

We have obtained recurrence relations for the CG and Racah coefficients of the quantum algebra $U_q(su_2)$. They generalize the corresponding formulae for the classical group SU(2). Recurrence relations for the Racah coefficients of $U_q(su_2)$ are obtained from classical ones by replacement of factorials $m!$ by q-factorials $[m]!$. Besides this replacement, in the recurrence formulae for the CG coefficients we have to add additional multipliers of the type q^s.

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